# Necessary and Sufficient Conditions for the Existence of Periodic Solutions of a Nicholson Type Delay System 

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#### Abstract

We consider a Nicholson type system for two species with mutualism and nonlinear harvesting terms. We give sufficient conditions for the existence of a positive periodic solution. We also provide a necessary condition; more precisely, we prove that if the harvesting rate is large enough, then 0 is a global attractor for the positive solutions and, in particular, positive periodic solutions cannot exist.


Keywords Nicholson type system • Nonlinear harvesting • Delay differential systems • Positive periodic solutions • Topological degree

Mathematics Subject Classification 34K13-92D25

## Introduction

The following system of delay differential equations was introduced in [5]:

$$
\left\{\begin{array}{l}
x_{1}^{\prime}(t)=-\delta_{1} x_{1}(t)+\beta_{1} x_{2}(t)+p_{1} x_{1}(t-\tau) e^{-\gamma_{1} x_{1}(t-\tau)}  \tag{1.1}\\
x_{2}^{\prime}(t)=-\delta_{2} x_{2}(t)+\beta_{2} x_{1}(t)+p_{2} x_{2}(t-\tau) e^{-\gamma_{2} x_{2}(t-\tau)} .
\end{array}\right.
$$

Here, $\delta_{i}, \beta_{i}, p_{i}, \gamma_{i}$ and $\tau$ are positive constants for $i=1,2$, with initial data $x_{i}(s)=$ $\phi_{i}(s), s \in[-\tau, 0], \phi_{i}(0)>0$ where $\phi_{i} \in C([-\tau, 0],[0,+\infty))$ for $i=1,2$. Systems of this kind were used, for example, in models of marine protected areas and to describe the dynamics of the $B$-cells of the chronic lymphocytic leukemia.

[^0]In [6], Zhou considered a non-autonomous version of system (1.1) with $T$-periodic $\delta_{i}, \beta_{i}, p_{i}, \gamma_{i} \in C\left(\mathbb{R}, \mathbb{R}^{+}\right)$. Instead of a constant $\tau$, each equation included a time-dependent $T$ periodic delay $\tau_{i} \in C\left(\mathbb{R}, \mathbb{R}^{+}\right)$and a linear harvesting term $-h_{i}(t) x_{i}\left(t-\tau_{i}(t)\right)$ for some $T$ periodic $h_{i} \in C(\mathbb{R},[0,+\infty))$. Existence of a positive (i.e. with strictly positive coordinates) $T$-periodic solution was proven, assuming:
(Z1) $\Delta:=\delta_{1 *} \delta_{2 *}-\beta_{1}^{*} \beta_{2}^{*}>0$,
(Z2) $\left(\frac{p_{i}}{\delta_{i}+h_{i}}\right)_{*}>1$,
(Z3) $\bar{\delta}_{i}+\bar{h}_{i}>\bar{\beta}_{i}$,
(Z4) $p_{i *} e^{-\gamma_{i}^{*} A_{i} / e \Delta} \geq h_{i}^{*}$,
where for a $T$-periodic function $x \in C\left(\mathbb{R}, \mathbb{R}^{+}\right)$we denote

$$
x^{*}:=\max _{t \in \mathbb{R}}\{x(t)\}, \quad x_{*}:=\min _{t \in \mathbb{R}}\{x(t)\}, \quad \bar{x}:=\frac{1}{T} \int_{0}^{T} x(t) d t
$$

and the quantities $A_{1}, A_{2}$ are defined by

$$
A_{1}:=\delta_{2 *}\left(\frac{p_{1}}{\gamma_{1}}\right)^{*}+\beta_{1}^{*}\left(\frac{p_{2}}{\gamma_{2}}\right)^{*}, \quad A_{2}:=\beta_{2}^{*}\left(\frac{p_{1}}{\gamma_{1}}\right)^{*}+\delta_{1 *}\left(\frac{p_{2}}{\gamma_{2}}\right)^{*} .
$$

In this work, we consider the case of a Nicholson type system for two species with mutualism and non-delayed nonlinear harvesting terms. For simplicity, we shall consider only the case $\gamma_{i} \equiv 1$, although the results in the present paper can be easily extended for arbitrary positive $T$-periodic functions $\gamma_{i}$. Setting $f(x):=x e^{-x}$, the system under study reads:

$$
\left\{\begin{array}{l}
x_{1}^{\prime}(t)=-\delta_{1}(t) x_{1}(t)+\beta_{1}(t) x_{2}(t)+p_{1}(t) f\left(x_{1}\left(t-\tau_{1}(t)\right)\right)-H_{1}\left(t, x_{1}(t)\right)  \tag{1.2}\\
x_{2}^{\prime}(t)=-\delta_{2}(t) x_{2}(t)+\beta_{2}(t) x_{1}(t)+p_{2}(t) f\left(x_{2}\left(t-\tau_{2}(t)\right)\right)-H_{2}\left(t, x_{2}(t)\right)
\end{array}\right.
$$

As before, $\delta_{i}, \beta_{i}, p_{i}, \tau_{i} \in C\left(\mathbb{R}, \mathbb{R}^{+}\right)$are $T$-periodic and $H_{i} \in C\left(\mathbb{R} \times \mathbb{R}^{+}, \mathbb{R}^{+}\right)$are $T$-periodic in $t$ for $i=1,2$. Under appropriate conditions we shall prove, using topological degree theory, the existence of at least one positive $T$-periodic solution. More precisely:

Theorem 1.1 Assume that the limits

$$
H_{i, s u p}^{0}(t):=\limsup _{x \rightarrow 0^{+}} \frac{H_{i}(t, x)}{x}
$$

and

$$
H_{i, i n f}^{\infty}(t):=\liminf _{x \rightarrow+\infty} \frac{H_{i}(t, x)}{x}
$$

are uniform in $t$ for $i=1,2$ and

$$
\begin{gather*}
\delta_{i}(t)+H_{i, \text { sup }}^{0}(t)<\beta_{i}(t)+p_{i}(t)  \tag{1.3a}\\
\delta_{i}(t)+H_{i, i n f}^{\infty}(t)>\beta_{i}(t) \tag{1.3b}
\end{gather*}
$$

for all $t \in \mathbb{R}, i=1,2$. Then the system (1.2) admits at least one positive $T$-periodic solution.

Moreover, we shall prove that the first condition of the previous theorem is, in some sense, expectable. With this aim, observe that (1.3a) implies that if $x>0$ is small enough then $\delta_{i}(t)+\frac{H_{i}(t, x)}{x}<\beta_{i}(t)+p_{i}(t)$ for all $t$. As we shall see, a necessary condition for the existence of positive $T$-periodic solutions is that the latter inequality holds for some $x>0$ and some $t$. In fact, we shall prove a bit more: if the inequality is reversed for all $t$ and all $x>0$, then the trivial equilibrium is a global attractor for the positive solutions.

Theorem 1.2 Let $\mathbf{x}:=\left(x_{1}, x_{2}\right)$ be a positive solution of (1.2) defined on $\left[t_{0},+\infty\right)$ for some $t_{0}$. Furthermore, assume that

$$
\begin{equation*}
\delta_{i}(t)+\frac{H_{i}(t, x)}{x} \geq \beta_{i}(t)+p_{i}(t) \quad \text { for all } t, \text { all } x>0 \text { and } i=1,2 . \tag{1.4}
\end{equation*}
$$

Then $\mathbf{x}(t) \rightarrow(0,0)$ as $t \rightarrow+\infty$. In particular, the problem does not admit positive $T$ periodic solutions.

As a corollary, we deduce that the solution of the initial value problem with positive data is positive, globally defined and tends to 0 as $t \rightarrow+\infty$. To this end, assume that $H_{i}$ is locally Lipschitz in $x$ and can be extended continuously to $\mathbb{R} \times[0,+\infty)$ as $H_{i}(\cdot, 0) \equiv 0$.

Corollary 1.3 Assume that $H_{i}$ is locally Lipschitz in its second variable and that $H_{i}(\cdot, 0) \equiv$ 0 . If $\phi_{i}:\left[-\tau_{i}^{*}, 0\right] \rightarrow \mathbb{R}^{+}$is continuous, then (1.2) with initial condition $x_{i}=\phi_{i}$ on $\left[-\tau_{i}^{*}, 0\right]$ has a unique solution $\mathbf{x}=\left(x_{1}, x_{2}\right)$, which is globally defined and positive. If furthermore (1.4) holds, then $\mathbf{x}(t) \rightarrow(0,0)$ as $t \rightarrow+\infty$.

## Existence of a Positive $\boldsymbol{T}$-Periodic Solution

## An Abstract Continuation Theorem

Let $C_{T} \subset C\left(\mathbb{R}, \mathbb{R}^{2}\right)$ be the Banach space of $T$-periodic continuous vector functions and consider the interior of the positive cone of $C_{T}$, namely

$$
X:=\left\{\mathbf{x}:=\left(x_{1}, x_{2}\right) \in C_{T}: x_{i}(t)>0 \text { for all } t \in \mathbb{R}, i=1,2\right\} .
$$

Define the operators

$$
L: X \cap C^{1}\left(\mathbb{R}, \mathbb{R}^{2}\right) \rightarrow C_{T}, \quad \Phi:=\left(\phi_{1}, \phi_{2}\right): X \rightarrow C_{T}
$$

in the following way:

$$
\begin{aligned}
L(\mathbf{x}) & :=\mathbf{x}^{\prime}, \\
\phi_{1}\left(x_{1}, x_{2}\right)(t) & :=-\delta_{1}(t) x_{1}(t)+\beta_{1}(t) x_{2}(t)+p_{1}(t) f\left(x_{1_{\tau_{1}}}(t)\right)-H_{1}\left(t, x_{1}(t)\right),
\end{aligned}
$$

and

$$
\phi_{2}\left(x_{1}, x_{2}\right)(t):=-\delta_{2}(t) x_{2}(t)+\beta_{2}(t) x_{1}(t)+p_{2}(t) f\left(x_{2 \tau_{2}}(t)\right)-H_{2}\left(t, x_{2}(t)\right) .
$$

For convenience, we shall employ the notation

$$
\mathbf{x}(t):=\left(x_{1}(t), x_{2}(t)\right), \quad x_{i \tau_{i}}(t):=x_{i}\left(t-\tau_{i}(t)\right) .
$$

Identifying the subspace of constant functions of $X$ with $\mathbb{R}^{2}$, we may set the mapping $g$ : $\mathbb{R}^{+} \times \mathbb{R}^{+} \rightarrow \mathbb{R}^{2}$ given by

$$
g(\mathbf{x})=-\overline{\Phi(\mathbf{x})}=-\frac{1}{T} \int_{0}^{T} \Phi(\mathbf{x})(t) d t
$$

The following continuation theorem can be readily adapted from the results in [2]:
Theorem 2.1 (Continuation theorem) Let $\bar{\Omega} \subset X$, with $\Omega$ open and bounded such that the following conditions are satisfied:
(H1) The equation $L(\mathbf{x})=\lambda \Phi(\mathbf{x})$ has no solutions on $\partial \Omega \cap C^{1}\left(\mathbb{R}, \mathbb{R}^{2}\right)$ for $\lambda \in(0,1)$.
(H2) $g$ does not vanish on $\partial \Omega \cap \mathbb{R}^{2}$ and $d_{B}\left(g, \Omega \cap \mathbb{R}^{2}, 0\right) \neq 0$, where $d_{B}$ denotes the Brouwer degree.
Then problem (1.2) has at least one solution $\mathbf{x} \in \bar{\Omega}$.

## A Priori Bounds

The following lemma shall be the key of the proof of our existence theorem. It is clear that if $\mathbf{x} \in X$ then there exist positive constants $\varepsilon<R$ such that $\varepsilon<x_{i}(t)<R$ for all $t$ and $i=1,2$. As we shall see, these constants may be chosen independently of $\mathbf{x}$ and $\lambda$ for the solutions of the problem $\mathbf{x}^{\prime}=\lambda \Phi(\mathbf{x})$ with $0<\lambda<1$, namely:

Lemma 2.2 Assume that the hypotheses of Theorem 1.1 are satisfied. Then there exist $\varepsilon_{0}, R_{0}>0$ such that if $\mathbf{x} \in X$ satisfies $\mathbf{x}^{\prime}=\lambda \Phi(\mathbf{x})$ with $x_{i}>0$ for $i=1,2$ and $\lambda \in(0,1)$ then

$$
\varepsilon_{0}<x_{i}(t)<R_{0} \quad \text { for all } t \in \mathbb{R}, i=1,2 .
$$

Proof Let $\mathbf{x}=\left(x_{1}, x_{2}\right) \in X$ satisfy $\mathbf{x}^{\prime}=\lambda \Phi(\mathbf{x})$ with $x_{i}>0$ for $i=1,2$ and $\lambda \in(0,1)$, and set $R$ as the maximum value between $x_{1}^{*}$ and $x_{2}^{*}$. Without loss of generality, we may assume for example that $x_{1}\left(t^{*}\right)=R$ for some $t^{*} \in[0, T]$.

From the first equation of the system, it follows that $\phi_{1}\left(x_{1}, x_{2}\right)\left(t^{*}\right)=0$, so

$$
\delta_{1}\left(t^{*}\right) x_{1}\left(t^{*}\right)+H_{1}\left(t^{*}, x_{1}\left(t^{*}\right)\right)=\beta_{1}\left(t^{*}\right) x_{2}\left(t^{*}\right)+p_{1}\left(t^{*}\right) f\left(x_{1 \tau_{1}}\left(t^{*}\right)\right) .
$$

Since $x_{2}\left(t^{*}\right) \leq R$ and $f\left(x_{1 \tau_{1}}\left(t^{*}\right)\right) \leq \frac{1}{e}$, we deduce:

$$
\begin{aligned}
R\left(\delta_{1}\left(t^{*}\right)+\frac{H_{1}\left(t^{*}, R\right)}{R}\right) & =\beta_{1}\left(t^{*}\right) x_{2}\left(t^{*}\right)+p_{1}\left(t^{*}\right) f\left(x_{1 \tau_{1}}\left(t^{*}\right)\right) \\
& \leq \beta_{1}\left(t^{*}\right) R+\frac{p_{1}^{*}}{e} .
\end{aligned}
$$

Hence

$$
R\left(\delta_{1}\left(t^{*}\right)+\frac{H_{1}\left(t^{*}, R\right)}{R}-\beta_{1}\left(t^{*}\right)\right) \leq \frac{p_{1}^{*}}{e} \leq \frac{p^{*}}{e},
$$

where $p^{*}:=\max \left\{p_{1}^{*}, p_{2}^{*}\right\}$. From (1.3b), there exist constants $\gamma, \tilde{R}>0$ such that

$$
\delta_{i}(t)+\frac{H_{i}(t, x)}{x}>\beta_{i}(t)+\gamma
$$

for all $x \geq \tilde{R}$ and all $t$. Thus,

$$
x_{i}(t)<\max \left\{\tilde{R}, \frac{p^{*}}{e \gamma}\right\}:=R_{0} \quad \text { for all } t \in \mathbb{R} \text { and } i=1,2 .
$$

In an analogous way, let $\varepsilon:=\min \left\{x_{1 *}, x_{2 *}\right\}$ and suppose for example that $x_{1}\left(t_{*}\right)=\varepsilon$ for some $t_{*}$, then

$$
\delta_{1}\left(t_{*}\right) x_{1}\left(t_{*}\right)+H_{1}\left(t_{*}, x_{1}\left(t_{*}\right)\right)=\beta_{1}\left(t_{*}\right) x_{2}\left(t_{*}\right)+p_{1}\left(t_{*}\right) f\left(x_{1 \tau_{1}}\left(t_{*}\right)\right) .
$$

We may assume that $R_{0} \geq 1$ and set $R_{1}$ as the unique value in $[0,1]$ such that $f\left(R_{1}\right)=$ $f\left(R_{0}\right)$. Moreover, we may assume that $\varepsilon \leq R_{1}$ since otherwise there is nothing to prove. Thus, $\varepsilon \leq x_{1}\left(t_{*}-\tau_{1}\left(t_{*}\right)\right) \leq R_{0}$ and

$$
f\left(x_{1 \tau_{1}}\left(t_{*}\right)\right) \geq f(\varepsilon),
$$

since $f$ is increasing in $[0,1]$ and decreasing in $[1,+\infty)$. Hence

$$
\begin{aligned}
\varepsilon\left(\delta_{1}\left(t_{*}\right)+\frac{H_{1}\left(t_{*}, \varepsilon\right)}{\varepsilon}\right) & =\beta_{1}\left(t_{*}\right) x_{2}\left(t_{*}\right)+p_{1}\left(t_{*}\right) f\left(x_{1 \tau_{1}}\left(t_{*}\right)\right) \\
& \geq \beta_{1}\left(t_{*}\right) \varepsilon+p_{1}\left(t_{*}\right) f(\varepsilon) \\
& =\varepsilon\left(\beta_{1}\left(t_{*}\right)+p_{1}\left(t_{*}\right) e^{-\varepsilon}\right)
\end{aligned}
$$

and

$$
\delta_{1}\left(t_{*}\right)+\frac{H_{1}\left(t_{*}, \varepsilon\right)}{\varepsilon} \geq \beta_{1}\left(t_{*}\right)+p_{1}\left(t_{*}\right) e^{-\varepsilon} \geq \beta_{1}\left(t_{*}\right)+p_{1}\left(t_{*}\right)-p^{*}\left(1-e^{-\varepsilon}\right) .
$$

Using (1.3a), we deduce the existence of $\eta, \tilde{\varepsilon}>0$ such that

$$
\delta_{i}(t)+\frac{H_{i}(t, x)}{x}<\beta_{i}(t)+p_{i}(t)-\eta
$$

for all $x \in(0, \tilde{\varepsilon}]$ and all $t$. Without loss of generality we may assume that $\eta<p^{*}$; thus,

$$
x_{i}(t)>\min \left\{\tilde{\varepsilon},-\ln \left(1-\eta / p^{*}\right)\right\}:=\varepsilon_{0} \text { for all } t \in \mathbb{R} \text { and } i=1,2 .
$$

Proof of Theorem 1.1 Based on Theorem 2.1, our aim consists in finding an open and bounded set $\Omega$ with $\bar{\Omega} \subset X$ such that (H1) and (H2) are verified.

From Lemma 2.2, if we set

$$
\Omega:=\left\{\mathbf{x} \in X: \varepsilon<x_{i}<R, i=1,2\right\}
$$

then (H1) holds for arbitrary positive constants $\varepsilon \leq \varepsilon_{0}$ and $R \geq R_{0}$, where $\varepsilon_{0}$ and $R_{0}$ are as in Lemma 2.2.

Thus, it suffices to consider the rectangle

$$
\Omega \cap \mathbb{R}^{2}=\left\{\mathbf{x} \in \mathbb{R}^{2}: \varepsilon<x_{i}<R, \quad i=1,2\right\}:=\mathcal{R}
$$

and prove that $g$ does not vanish at $\partial \mathcal{R}$ and $\operatorname{deg}(g, \mathcal{R}, 0) \neq 0$.
In the first place, observe that

$$
\begin{aligned}
g_{1}\left(x_{1}, x_{2}\right) & =\bar{\delta}_{1} x_{1}-\bar{\beta}_{1} x_{2}-\bar{p}_{1} x_{1} e^{-x_{1}}+\frac{x_{1}}{T} \int_{0}^{T} \frac{H_{1}\left(t, x_{1}\right)}{x_{1}} d t \\
& =\frac{x_{1}}{T} \int_{0}^{T}\left(\delta_{1}(t)-p_{1}(t) e^{-x_{1}}+\frac{H_{1}\left(t, x_{1}\right)}{x_{1}}\right) d t-\bar{\beta}_{1} x_{2} .
\end{aligned}
$$

In particular, if $x_{1}=\varepsilon \leq x_{2} \leq R$ then

$$
\begin{aligned}
g_{1}\left(\varepsilon, x_{2}\right) & =\frac{\varepsilon}{T} \int_{0}^{T}\left(\delta_{1}(t)-p_{1}(t) e^{-\varepsilon}+\frac{H_{1}(t, \varepsilon)}{\varepsilon}\right) d t-\bar{\beta}_{1} x_{2} \\
& \leq \frac{\varepsilon}{T} \int_{0}^{T}\left(\delta_{1}(t)-p_{1}(t) e^{-\varepsilon}+\frac{H_{1}(t, \varepsilon)}{\varepsilon}\right) d t-\bar{\beta}_{1} \varepsilon \\
& =\frac{\varepsilon}{T} \int_{0}^{T}\left(\delta_{1}(t)-p_{1}(t) e^{-\varepsilon}-\beta_{1}(t)+\frac{H_{1}(t, \varepsilon)}{\varepsilon}\right) d t=g_{1}(\varepsilon, \varepsilon) .
\end{aligned}
$$

By (1.3a), we deduce, for all $t \in \mathbb{R}$,

$$
\limsup _{\varepsilon \rightarrow 0^{+}} \frac{H_{1}(t, \varepsilon)}{\varepsilon}<-\delta_{1}(t)+p_{1}(t)+\beta_{1}(t)=\lim _{\varepsilon \rightarrow 0^{+}}\left(-\delta_{1}(t)+p_{1}(t) e^{-\varepsilon}+\beta_{1}(t)\right)
$$

and hence

$$
\limsup _{\varepsilon \rightarrow 0^{+}}\left(\frac{H_{1}(t, \varepsilon)}{\varepsilon}+\delta_{1}(t)-p_{1}(t) e^{-\varepsilon}-\beta_{1}(t)\right)<0
$$

uniformly in $t$. Thus, if $\varepsilon \in\left(0, \varepsilon_{0}\right)$ is small enough, then

$$
g_{1}\left(\varepsilon, x_{2}\right) \leq g_{1}(\varepsilon, \varepsilon)<0, \quad \text { for all } \varepsilon \leq x_{2} \leq R .
$$

Now suppose $\varepsilon \leq x_{2} \leq R=x_{1}$, then

$$
\begin{aligned}
g_{1}\left(R, x_{2}\right) & =\frac{R}{T} \int_{0}^{T}\left(\delta_{1}(t)-p_{1}(t) e^{-R}+\frac{H_{1}(t, R)}{R}\right) d t-\bar{\beta}_{1} x_{2} \\
& \geq \frac{R}{T} \int_{0}^{T}\left(\delta_{1}(t)-p_{1}(t) e^{-R}+\frac{H_{1}(t, R)}{R}\right) d t-\bar{\beta}_{1} R \\
& =\frac{R}{T} \int_{0}^{T}\left(\delta_{1}(t)-p_{1}(t) e^{-R}-\beta_{1}(t)+\frac{H_{1}(t, R)}{R}\right) d t=g_{1}(R, R) .
\end{aligned}
$$

Using (1.3b), it is seen that

$$
\liminf _{R \rightarrow+\infty} \frac{H_{1}(t, R)}{R}>-\delta_{1}(t)+\beta_{1}(t)=\lim _{R \rightarrow+\infty}\left(-\delta_{1}(t)+p_{1}(t) e^{-R}+\beta_{1}(t)\right)
$$

and thus

$$
\liminf _{R \rightarrow+\infty}\left(\frac{H_{1}(t, R)}{R}+\delta_{1}(t)-p_{1}(t) e^{-R}-\beta_{1}(t)\right)>0
$$

uniformly in $t$.
We conclude that if $R>R_{0}$ is large enough then

$$
g_{1}\left(R, x_{2}\right) \geq g_{1}(R, R)>0, \quad \text { for all } \varepsilon \leq x_{2} \leq R .
$$

In the same way, it is verified that if $R \geq R_{0}$ is large enough and $\varepsilon \in\left(0, \varepsilon_{0}\right)$ is small enough then

$$
g_{2}\left(x_{1}, \varepsilon\right)<0<g_{2}\left(x_{1}, R\right), \quad \text { for all } \varepsilon \leq x_{1} \leq R .
$$

Hence $d_{B}(g, \mathcal{R}, 0)$ is well defined and different from zero, so (H2) is satisfied and the proof is complete.

Remark 2.1 It is worth noticing that the asymptotic condition (1.3b) is required due to the mutualism. In the scalar case (see e.g. [1]), no condition at $+\infty$ is assumed, since large populations are self-regulated by the action of the nonlinearity $f$.

Remark 2.2 Interestingly, the conditions in Theorem 1.1 do not seem to suffice for the case of delayed harvesting terms, namely $-H_{i}\left(t, x_{i}\left(t-\tau_{i}(t)\right)\right)$. This fact partially explains the more restrictive set of conditions mentioned above, obtained in [6] for the linear case $H_{i}\left(t, x_{i}(t-\right.$ $\left.\left.\tau_{i}(t)\right)\right)=h_{i}(t) x_{i}\left(t-\tau_{i}(t)\right)$. The preceding continuation theorem can be used to extend the latter conditions for the nonlinear case. For example, observe that (Z1) implies, for some $i$, that $\delta_{i}(t)>\beta_{i}(t)$ for all $t$; if for simplicity we assume this is true for both $i=1,2$, then upper bounds are readily obtained as in Lemma 2.2 with $R:=\frac{p^{*}}{e(\delta-\beta)_{*}}$, where $(\delta-\beta)_{*}:=$ $\min \left\{\left(\delta_{1}-\beta_{1}\right)_{*},\left(\delta_{2}-\beta_{2}\right)_{*}\right\}$. Thus, existence of positive $T$-periodic solutions can be proven, provided that
(1) (1.3a) holds,
(2) There exists $\eta>0$ such that the function $\varphi_{i}: \mathbb{R} \times \mathbb{R}^{+} \rightarrow \mathbb{R}$ given by $\varphi_{i}(t, x):=$ $p_{i}(t) f(x)-H_{i}(t, x)$ is nondecreasing in $x$ for $0<x<\eta$.
(3) $\varphi_{i}(t, x)>0$ for $0<x \leq R$.

Indeed, if as in Lemma 2.2 we let $\varepsilon:=\min \left\{x_{1 *}, x_{2 *}\right\}$ and suppose for example that $x_{1}\left(t_{*}\right)=\varepsilon$ for some $t_{*}$, then it is deduced that either

$$
\left(\delta_{1}\left(t_{*}\right)-\beta_{1}\left(t_{*}\right)\right) \varepsilon \geq \inf _{\eta \leq x \leq R} p_{1}\left(t_{*}\right) f(x)-H_{1}\left(t_{*}, x\right)
$$

or

$$
\left(\delta_{1}\left(t_{*}\right)-\beta_{1}\left(t_{*}\right)\right) \varepsilon \geq p_{1}\left(t_{*}\right) f(\varepsilon)-H_{1}\left(t_{*}, \varepsilon\right)
$$

and the proof follows as before.
It is worthy noticing that, for the linear case, (1.3c) and (1.3d) are satisfied if $p_{i}(t) e^{-R}>$ $h_{i}(t)$, a condition comparable to (Z4). Moreover, in this situation (1.3a) simply reads $\delta_{i}(t)+$ $h_{i}(t)<\beta_{i}(t)+p_{i}(t)$ for all $t$, which is obviously weaker than (Z2).

## Necessary Conditions. Global Stability of the Trivial Equilibrium

In this section, we shall give a proof of Theorem 1.2 and Corollary 1.3. Specifically, we shall prove that if (1.4) holds, then all positive solutions defined for $t \geq t_{0}$ tend to 0 as $t \rightarrow+\infty$. As a consequence, it is seen that if $H_{i}$ satisfy a Lipschitz condition for $i=1,2$, then the solutions of the initial value problem with positive data are globally defined, positive and tend to 0 as $t \rightarrow+\infty$.

Proof of Theorem 1.2 Let $\mathbf{x}$ be a positive solution defined for $t \geq t_{0}$. For convenience, let us employ the following notation: if $i=1$ or 2 , then denote by $j$ the remaining element of the set $\{1,2\}$. Also, we define $e_{1}:=f(1)$ and $e_{n+1}:=f\left(e_{n}\right)$. Clearly the sequence $\left\{e_{n}\right\}$ is strictly decreasing and tends to 0 as $n \rightarrow \infty$.

Due to (1.4), it is seen that

$$
\begin{equation*}
x_{i}^{\prime}(t) \leq p_{i}(t)\left(f\left(x_{i \tau_{i}}(t)\right)-x_{i}(t)\right)+\beta_{i}(t)\left(x_{j}(t)-x_{i}(t)\right) . \tag{3.1}
\end{equation*}
$$

Hence, if $x_{i}(t) \geq e_{1}$ and $x_{j}(t) \leq x_{i}(t)$, then $x_{i}^{\prime}(t) \leq 0$. Furthermore, if one of the first inequalities is strict then the latter one is also strict.

Claim There exists $t_{1}$ such that $x_{i}(t) \leq e_{1}$ for all $t \geq t_{1}$.
We shall prove the claim in two steps. Firstly, we shall prove that $x_{i}\left(t_{1}\right) \leq e_{1}$ for some $t_{1}$ and $i=1,2$. Next, it shall be seen that $x_{i}(t) \leq e_{1}$ for all $t \geq t_{1}$. With this aim, define the following sets:

$$
\begin{aligned}
R_{i} & :=\left\{\mathbf{x} \in \mathbb{R}^{+} \times \mathbb{R}^{+}: x_{i} \geq \max \left\{x_{j}, e_{1}\right\}\right\} \\
E & :=\left\{\mathbf{x} \in \mathbb{R}^{+} \times \mathbb{R}^{+}: x_{i} \leq e_{1} \text { for } i=1,2\right\}
\end{aligned}
$$

and

$$
D:=\left\{\mathbf{x} \in \mathbb{R}^{+} \times \mathbb{R}^{+}: x_{1}=x_{2}>e_{1}\right\}=R_{1} \cap R_{2} \backslash\left\{\left(e_{1}, e_{1}\right)\right\} .
$$

Suppose that $\mathbf{x}(t) \notin E$ for all $t$. If there exists $t_{1}$ such that $\mathbf{x}(t) \in R_{i}$ for $t \geq t_{1}$, then $x_{i}$ is nonincreasing after $t_{1}$ and thus converges to some value $a \geq e_{1}$. Hence $f\left(x_{i \tau_{i}}(t)\right)-x_{i}(t) \rightarrow$ $f(a)-a<0$ and $x_{i}^{\prime}(t) \leq p_{i}(t)\left(f\left(x_{i \tau_{i}}(t)\right)-x_{i}(t)\right) \leq c<0$ for $t$ large enough, a contradiction. From now on, we may assume that $\mathbf{x}(t)$ does not remain in $R_{i}$ and, in particular, there exists a sequence $t_{n} \rightarrow+\infty$ such that $\mathbf{x}\left(t_{n}\right) \in D$ for all $n$.

Observe that if $\mathbf{x}(t) \in D$ then $x_{i}^{\prime}(t) \leq p_{i}(t) f\left(x_{i \tau_{i}}(t)\right)-x_{i}(t)<0$. On the other hand, if $\mathbf{x}(t) \in R_{i}^{\circ}$ for all $t \in(r, s)$ then $x_{i}(s)<x_{i}(t)<x_{i}(r)$ and $x_{j}(t)<x_{i}(t)$ for $t \in(r, s)$. This yields the following conclusion: if $\mathbf{x}(s) \in D$ then $x_{i}(t)<x_{i}(s)$ for all $t>s$. In particular, $x_{i}\left(t_{n}\right)$ is strictly decreasing and converges to some value $a \geq e_{1}$.

We claim there exists a constant $c<0$ independent of $t$ such that if $\mathbf{x}(t) \in R_{i}$ then $x_{i}^{\prime}(t) \leq c$. Indeed, by inequality (3.1) it suffices to show that $f\left(x_{i \tau_{i}}(t)\right)-x_{i}(t) \leq C<0$ for some constant $C$ independent of $t$. There exist two cases:

1. If $a>e_{1}$ then $f\left(x_{i \tau_{i}}(t)\right)-x_{i}(t) \leq e_{1}-a<0$.
2. If $a=e_{1}$, then fix $s$ such that $x_{i}(r)<\frac{1}{2}$ for $r>s$. If $t>s+\tau_{i}^{*}$ then $f\left(x_{i \tau_{i}}(t)\right)-x_{i}(t) \leq$ $f\left(\frac{1}{2}\right)-e_{1}<0$. Otherwise, fix $n$ such that $t_{n} \geq s+\tau_{i}^{*}$. Then $x_{i}(t) \geq x_{i}\left(t_{n}\right)>e_{1}$ and thus $f\left(x_{i \tau_{i}}(t)\right)-x_{i}(t) \leq e_{1}-x_{i}\left(t_{n}\right)<0$.

Next, consider the closed set $\mathcal{C}:=\left\{t \geq t_{0}: \mathbf{x}(t) \in D\right\}$ and write

$$
\left[t_{0},+\infty\right) \backslash \mathcal{C}=\bigcup_{k=1}^{\infty}\left(a_{k}, b_{k}\right)
$$

where the intervals $\left(a_{k}, b_{k}\right)$ are nonempty and disjoint, with $\mathbf{x}\left(a_{k}\right), \mathbf{x}\left(b_{k}\right) \in D$ and $\mathbf{x}\left(\left[a_{k}, b_{k}\right]\right) \subset R_{i}$ for some $i$. Let us compute

$$
x_{i}\left(t_{n}\right)-x_{i}\left(t_{0}\right)=\int_{t_{0}}^{t_{n}} x_{i}^{\prime}(s) d s=\int_{\mathcal{C} \cap\left[t_{0}, t_{n}\right]} x_{i}^{\prime}(s) d s+\int_{\left[t_{0}, t_{n}\right] \backslash \mathcal{C}} x_{i}^{\prime}(s) d s .
$$

Recall that $x_{i}^{\prime}(t) \leq c$ over $\mathcal{C}$; thus, $\int_{\mathcal{C} \cap\left[t_{0}, t_{n}\right]} x_{i}^{\prime}(s) d s \leq c\left|\mathcal{C} \cap\left[t_{0}, t_{n}\right]\right|$. On the other hand,

$$
\int_{a_{k}}^{b_{k}} x_{j}^{\prime}(t) d t=x_{j}\left(b_{k}\right)-x_{j}\left(a_{k}\right)=x_{i}\left(b_{k}\right)-x_{i}\left(a_{k}\right)=\int_{a_{k}}^{b_{k}} x_{i}^{\prime}(t) d t \leq c\left(b_{k}-a_{k}\right)
$$

and we deduce that $\int_{\left[t_{0}, t_{n}\right] \backslash \mathcal{C}} x_{i}^{\prime}(s) d s \leq c\left|\left[t_{0}, t_{n}\right] \backslash \mathcal{C}\right|$. Hence

$$
x_{i}\left(t_{n}\right)-x_{i}\left(t_{0}\right) \leq c\left|\mathcal{C} \cap\left[t_{0}, t_{n}\right]\right|+c\left|\left[t_{0}, t_{n}\right] \backslash \mathcal{C}\right|=c\left(t_{n}-t_{0}\right) \rightarrow-\infty,
$$

a contradiction.
For the second step, let us prove that if $\mathbf{x}(s) \in E$ then $\mathbf{x}(t) \in E$ for all $t \geq s$. From (3.1), it is clear that $\mathbf{x}$ cannot abandon $E$ through any of the segments $\left\{\mathbf{x}: x_{i}=e_{1}>x_{j}>0\right\}$ for $i=1,2$. Moreover, since $x_{i}^{\prime}(t)<0$ when $\mathbf{x}(t) \in D$, it follows that $\mathbf{x}$ cannot abandon $E$ through the point $\left(e_{1}, e_{1}\right)$ either.

Next, we may use (3.1) again to deduce, for $t \geq t_{1}+\tau_{i}^{*}$, that if $x_{i}(t) \geq e_{2}$ and $x_{j}(t) \leq x_{i}(t)$, then $x_{i}^{\prime}(t) \leq 0$; furthermore, if one of the first inequalities is strict then the latter one is also strict. Thus, repeating the previous procedure we obtain $t_{2}>t_{1}+\tau_{i}^{*}$ such that $x_{i}(t) \geq e_{2}$ for all $t \geq t_{2}$. Inductively, there exists an increasing sequence $\left\{t_{n}\right\}$ such that $x_{i}(t) \leq e_{n}$ for all $t \geq t_{n}$, and the proof is complete.

Proof of Corollary 1.3 Assume that $H_{i}(\cdot, 0) \equiv 0$ and observe that if $x_{1}(t), x_{2}(t)>0$ for $t<t_{i}$ and $x_{i}\left(t_{i}\right)=0$ then

$$
0 \geq x_{i}^{\prime}\left(t_{i}\right)=\beta_{i}\left(t_{i}\right) x_{j}\left(t_{i}\right)+p_{i}(t) f\left(x_{i \tau_{i}}\left(t_{i}\right)\right)-H_{i}\left(t_{i}, 0\right)>0,
$$

a contradiction.
On the other hand,

$$
x_{i}^{\prime}(t) \leq \frac{p_{i}(t)}{e}+\beta_{i}(t) x_{j}(t)
$$

and by standard results (see e.g. [3,4]), we deduce that the initial value problem has a unique solution $\mathbf{x}$, which is defined for all $t \geq 0$. The conclusion thus follows from Theorem 1.2.

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