POSITIVE T-PERIODIC SOLUTIONS FOR A NICHOLSON TYPE SYSTEM.

Amster Pablo[†‡] and Déboli Alberto[‡]

[†]*Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina. pamster@dm.uba.ar*
[‡]*Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires. Ciudad Universitaria, Pabellón I, 1428 Buenos Aires, Argentina. adeboli@dm.uba.ar*

Abstract: In this work, we consider the case of a Nicholson type system for two species with mutualism terms. Under appropriate conditions we prove, using topological degree theory, the existence of at least one positive T-periodic solution.

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1 INTRODUCTION.

The dynamics of the following system was studied in [2]:

$$\begin{cases} x_1'(t) = -\delta_1 x_1(t) + \beta_1 x_2(t) + p_1 x_1(t-\tau_1)e^{-a_1 x_1(t-\tau_1)} \\ \\ x_2'(t) = -\delta_2 x_2(t) + \beta_2 x_1(t) + p_2 x_2(t-\tau_2)e^{-a_2 x_1(t-\tau_2)} \end{cases} \tag{1}$$

with initial data $x_i(s) = \phi_i(s),\ s \in [-\tau, 0],\ \phi_i(0) > 0$ where $\phi_i \in C([-\tau, 0], [0, +\infty))$ for $i = 1, 2$. Here, $\delta_i, \beta_i, p_i, a_i$ and τ are positive constants for $i = 1, 2$, Models of this kind were used, for example, to describe the dynamics of the B-cells of the chronic lymphocytic leukemia.

In [3], Zhou considered a non-autonomous version of system (1) with T-periodic $\delta_i, \beta_i, p_i, \tau_i \in C(\mathbb{R}, \mathbb{R}^+)$, which included in each equation a linear harvesting term $H_i := h_i(t)x_i(t - \tau_i(t))$ for some T-periodic $h_i \in C(\mathbb{R}, [0, +\infty))$. Under appropriate conditions, existence of a positive T-periodic solution was proven.

In this work, we consider the case of a Nicholson type system for two species with mutualism terms

$$\begin{cases} x_1'(t) = -\delta_1(t)x_1(t) + \beta_1(t)x_2(t) + p_1(t)f(x_1(t-\tau_1(t))) - H_1(t, x_1(t)) \\ \\ x_2'(t) = -\delta_2(t)x_2(t) + \beta_2(t)x_1(t) + p_2(t)f(x_2(t-\tau_2(t))) - H_2(t, x_2(t)). \end{cases} \tag{2}$$

Here, the functions $\delta_i, \beta_i, p_i, \tau_i \in C(\mathbb{R}, \mathbb{R}^+)$ are T-periodic, $f(x) = xe^{-x}$ and $H_i \in C(\mathbb{R} \times \mathbb{R}^+, \mathbb{R}^+)$ are T-periodic in t for $i = 1, 2$. Under appropriate conditions we shall prove, using topological degree theory, the existence of at least one positive T-periodic solution.

Our main theorem reads as follows.

Theorem 1.1 *Assume that the limits*

$$H_{i,sup}(t) := \limsup_{x \to 0^+} \frac{H_i(t, x)}{x}$$

and

$$H_{i,inf}(t) := \liminf_{x \to +\infty} \frac{H_i(t, x)}{x}$$

are uniform in t for $i = 1, 2$ and

$$\delta_i(t) + H_{i,sup}(t) < \beta_i(t) + p_i(t) \tag{3a}$$

$$\delta_i(t) + H_{i,inf}(t) > \beta_i(t) + p_i(t) \tag{3b}$$

for all $t \in \mathbb{R}$, $i = 1, 2$. Then the system (2) admits at least one positive T-periodic solution.

2 Existence of solutions

2.0.1 An abstract continuation theorem

Consider the Banach space

$$X := \{\mathbf{x} := (x_1, x_2) \in C(\mathbb{R}, \mathbb{R}^2) : x_i(t+T) = x_i(t), i = 1, 2\}$$

equipped with the norm $||\mathbf{x}|| = \max\{|x_i|_\infty, i = 1, 2\}$, where

$$|x_i|_\infty := \max_{t\in[0,T]} \{|x_i(t)|\},\ i = 1, 2$$

and the operators

$$L : X \cap C^1(\mathbb{R}, \mathbb{R}^2) \subset X \to X, \quad \Phi := (\phi_1, \phi_2) :\ X \to X$$

are defined in the following way:

$$L(\mathbf{x}) = \mathbf{x}'$$

$$\phi_1(x_1, x_2)(t) := -\delta_1(t)x_1(t) + \beta_1(t)x_2(t) + p_1(t)f(x_{1_{\tau_1}}(t)) - H_1(t, x_1(t)))$$

and

$$\phi_2(x_1, x_2)(t) := -\delta_2(t)x_2(t) + \beta_2(t)x_1(t) + p_2(t)f(x_{2_{\tau_2}}(t)) - H_2(t, x_2(t))).$$

For convenience, we shall employ the notation

$$\overline{x} := \frac{1}{T}\int_0^T x(t)\, dt,\ \ \overline{\mathbf{x}} := (\overline{x}_1, \overline{x}_2), \quad x_{i\tau_i}(t) := x_i(t - \tau_i(t))$$

and set the mapping $g : \mathbb{R}^2 \to \mathbb{R}^2$ given by $g(\mathbf{x}) = -\overline{\Phi(\mathbf{x})}$.

The following continuation theorem was proven in [1]:

Theorem 2.1 (Continuation theorem) *Let $\Omega \subset X$ be open and bounded such that the following conditions are satisfied:*

(H1): The equation $L(\mathbf{x}) = \lambda\Phi(\mathbf{x})$ has no solutions on $\partial\Omega \cap Dom(L)$ for $\lambda \in (0, 1)$.

(H2): g does not vanish on $\partial\Omega \cap \mathbb{R}^2$ and $d_B(g, \Omega \cap \mathbb{R}^2, 0) \neq 0$, where d_B denotes the Brouwer degree.

Then problem (2) has at least one solution $\mathbf{x} \in \overline{\Omega} \subset X$ such that $x_i > 0$ for $i = 1, 2$.

2.0.2 A priori bounds

The following lemma shall be the key of the proof of our existence theorem.

Lemma 2.2 *Assume that the hypotheses of Theorem 1.1 are satisfied. Then there exist $\varepsilon_0, R_0 > 0$ such that if $\mathbf{x} \in X$ satisfies $\mathbf{x}' = \lambda\Phi(\mathbf{x})$ with $x_i > 0$ for $i = 1, 2$ and $\lambda \in (0, 1)$ then*

$$\varepsilon_0 < x_i(t) < R_0 \ \ \textit{for all } t \in \mathbb{R},\ i = 1, 2.$$

Proof.

Let $\mathbf{x} = (x_1, x_2) \in X$ satisfy $\mathbf{x}' = \lambda\Phi(\mathbf{x})$ with $x_i > 0$ for $i = 1, 2$ and $\lambda \in (0, 1)$, and set $R := \max\{x_1^*, x_2^*\}$ where $x^* := \max_{t\in\mathbb{R}}\{x(t)\}$. Without loss of generality, we may assume for example that $x_1(t^*) = R$ for some $t^* \in [0, T]$.

From the first equation of the system it is seen that $\phi_1(x_1(t^*), x_2(t^*)) = 0$, so

$$\delta_1(t^*)x_1(t^*) - H_1(t^*, x_1(t^*)) = \beta_1(t^*)x_2(t^*) + p_1(t^*)f(x_{1\tau_1}(t^*)).$$

Since $x_2(t^*) \le R$ and $f(x_{1\tau_1}(t^*)) \le \frac{1}{e}$, we deduce:

$$R\left(\delta_1(t^*) + \frac{H_1(t^*,R)}{R}\right) = \beta_1(t^*)x_2(t^*) + p_1(t^*)f(x_{1\tau_1}(t^*)) \le \beta_1(t^*)R + \frac{p_1^*}{e}.$$

Hence

$$R\left(\delta_1(t^*) + \frac{H_1(t^*,R)}{R} - \beta_1(t^*)\right) \le \frac{p_1^*}{e}$$

and using (3b) we conclude that there exists a constant $R_0 > 0$ such that

$$x_i(t) < R_0 \text{ for all } t \in \mathbb{R} \text{ and } i = 1,2.$$

In an analogous way, let $\varepsilon := \min\{x_{1*}, x_{2*}\}$ where $x_* := \min_{t\in\mathbb{R}}\{x(t)\}$ and suppose for example that $x_1(t_*) = \varepsilon$ for some t_*, then

$$\delta_1(t_*)x_1(t_*) - H_1(t_*, x_1(t_*)) = \beta_1(t_*)x_2(t_*) + p_1(t_*)f(x_{1\tau_1}(t_*)).$$

We may assume that $R_0 \ge 1$ and set R_1 as the unique value in $[0,1]$ such that $f(R_1) = f(R_0)$. Moreover, we may assume that $\varepsilon \le R_1$ since otherwise there is nothing to prove. Thus, $\varepsilon \le x_1(t_* - \tau_1(t_*)) \le R_0$ and

$$f(x_{1\tau_1}(t_*)) \ge f(\varepsilon),$$

since f is increasing in $[0,1]$ and decreasing in $[1,+\infty)$. Hence

$$\begin{aligned}&\varepsilon\left(\delta_1(t_*) + \frac{H_1(t_*,\varepsilon)}{\varepsilon}\right) = \beta_1(t_*)x_2(t_*) + p_1(t_*)f(x_{1\tau_1}(t_*)) \ge \\ &\beta_1(t_*)\varepsilon + p_1(t_*)f(\varepsilon) = \varepsilon\left(\beta_1(t_*) + p_1(t_*)e^{-\varepsilon}\right)\end{aligned}$$

and

$$\delta_1(t_*) + \frac{H_1(t_*,\varepsilon)}{\varepsilon} \ge \beta_1(t_*) + p_1(t_*)e^{-\varepsilon}.$$

Using (3a), we deduce the existence of $\varepsilon_0 > 0$ such that $\varepsilon_0 < x_i(t)$ for all $t \in \mathbb{R}$ and $i = 1,2$ ■

2.0.3 Proof of the existence theorem

In this section, we shall prove the existence of $\Omega \subset X$ open and bounded such that (H1) and (H2) of Theorem 2.1 are verified.

From the previous section, if we set

$$\Omega := \{\mathbf{x} \in X : \varepsilon < x_i < R,\ i = 1,2\}$$

then (H1) holds for arbitrary positive constants $\varepsilon \le \varepsilon_0$ and $R \ge R_0$, where ε_0 and R_0 are as in lemma 2.2.
Thus, it suffices to prove that the Brouwer degree of the function g is different from zero over the set

$$\Omega_0 := \{\mathbf{x} \in \mathbb{R}^2 : \ \varepsilon < x_i < R, \ \ i = 1,2\}.$$

In the first place, observe that

$$\begin{aligned}&g_1(x_1,x_2) = \overline{\delta}_1x_1 - \overline{\beta}_1x_2 - \overline{p}_1x_1e^{-x_1} + \frac{x_1}{T}\int_0^T \frac{H_1(t,x_1)}{x_1}dt = \\ &= \frac{x_1}{T}\int_0^T \left(\delta_1(t) - p_1(t)e^{-x_1} + \frac{H(t,x_1)}{x_1}\right)dt - \overline{\beta}_1x_2.\end{aligned}$$

In particular, if $x_1 = \varepsilon \le x_2 \le R$ then

$$
\begin{aligned}
&g_1(\varepsilon, x_2) = \overline{\delta}_1\varepsilon - \overline{\beta}_1 x_2 - \overline{p}_1\varepsilon e^{-\varepsilon} + \tfrac{\varepsilon}{T}\int_0^T \tfrac{H_1(t,\varepsilon)}{\varepsilon}dt = \\
&= \tfrac{\varepsilon}{T}\int_0^T \left(\delta_1(t) - p_1(t)e^{-\varepsilon} + \tfrac{H(t,\varepsilon)}{\varepsilon}\right)dt - \overline{\beta}_1 x_2 \leq \\
&\tfrac{\varepsilon}{T}\int_0^T \left(\delta_1(t) - p_1(t)e^{-\varepsilon} + \tfrac{H(t,\varepsilon)}{\varepsilon}\right)dt - \overline{\beta}_1 \varepsilon = \\
&\tfrac{\varepsilon}{T}\int_0^T \left(\delta_1(t) - p_1(t)e^{-\varepsilon} - \beta_1(t) + \tfrac{H(t,\varepsilon)}{\varepsilon}\right)dt = g_1(\varepsilon, \varepsilon).
\end{aligned}
$$

Hence by (3a) we deduce, for all $t \in \mathbb{R}$,

$$
\limsup_{\varepsilon \to 0^+} \frac{H(t,\varepsilon)}{\varepsilon} < -\delta_1(t) + p_1(t) + \beta_1(t) = \lim_{\varepsilon \to 0^+}(-\delta_1(t) + p_1(t)e^{-\varepsilon} + \beta_1(t))
$$

and thus

$$
\limsup_{\varepsilon \to 0^+} \left(\frac{H(t,\varepsilon)}{\varepsilon} + \delta_1(t) - p_1(t)e^{-\varepsilon} - \beta_1(t)\right) < 0
$$

uniformly en t.

Thus, if $\varepsilon \in (0, \varepsilon_0)$ is small enough, then

$$
g_1(\varepsilon, x_2) \leq g(\varepsilon, \varepsilon) < 0, \quad \text{for all } \varepsilon \leq x_2 \leq R.
$$

Now suppose $\varepsilon \leq x_2 \leq R = x_1$, then

$$
\begin{aligned}
&g_1(R, x_2) = \overline{\delta}_1 R - \overline{\beta}_1 x_2 - \overline{p}_1 R e^{-R} + \tfrac{R}{T}\int_0^T \tfrac{H_1(t,R)}{R}dt = \\
&= \tfrac{R}{T}\int_0^T \left(\delta_1(t) - p_1(t)e^{-R} + \tfrac{H(t,R)}{R}\right)dt - \overline{\beta}_1 x_2 \geq \\
&\tfrac{R}{T}\int_0^T \left(\delta_1(t) - p_1(t)e^{-R} + \tfrac{H(t,R)}{R}\right)dt - \overline{\beta}_1 R = \\
&\tfrac{R}{T}\int_0^T \left(\delta_1(t) - p_1(t)e^{-R} - \beta_1(t) + \tfrac{H(t,R)}{R}\right)dt = g_1(R, R).
\end{aligned}
$$

Using (3b), it is seen that

$$
\liminf_{R \to +\infty} \frac{H(t,R)}{R} < -\delta_1(t) + \beta_1(t) = \lim_{R \to +\infty}(-\delta_1(t) + p_1(t)e^{-R} + \beta_1(t))
$$

uniformly in t and thus

$$
\liminf_{R \to +\infty} \left(\frac{H(t,\varepsilon)}{\varepsilon} + \delta_1(t) - p_1(t)e^{-R} - \beta_1(t)\right) > 0
$$

uniformly in t.

We conclude that if $R > R_0$ is large enough then

$$
g_1(R, x_2) \geq g_1(R, R) > 0, \quad \text{for all } \varepsilon \leq x_2 \leq R.
$$

In the same way, it is verified that if $R \geq R_0$ is large enough and $\varepsilon \in (0, \varepsilon_0)$ is small enough then

$$
g_2(x_1, \varepsilon) < 0, \quad \text{for all } \varepsilon \leq x_1 \leq R
$$

and

$$
g_2(x_1, R) > 0, \quad \text{for all } \varepsilon \leq x_1 \leq R.
$$

Hence $d_B(g, \Omega \cap \mathbb{R}^2, 0) \neq 0$ is well defined and different from zero, so (H2) is satisfied and the proof is complete. ■

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